Teoretická mechanika – užitečné vzorce

(http://www.physics.muni.cz/~tomtyc/vzorce-mech.pdf)

- Úplná časová derivace veličiny A (např. teploty nebo složky rychlosti), která závisí na čase i na prostorových souřadnicích:

$$\begin{aligned}\frac{\mathrm{d}A(x,y,z,t)}{\mathrm{d}t} &= \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}\frac{\mathrm{dx}}{\mathrm{d}t} + \frac{\partial A}{\partial y}\frac{\mathrm{dy}}{\mathrm{d}t} + \frac{\partial A}{\partial z}\frac{\mathrm{dz}}{\mathrm{d}t} \\ &= \frac{\partial A}{\partial t} + \dot{x}\frac{\partial A}{\partial x} + \dot{y}\frac{\partial A}{\partial y} + \dot{z}\frac{\partial A}{\partial z} \\ &= \frac{\partial A}{\partial t} + (\dot{\vec{r}}\cdot\nabla)A\end{aligned}$$

- **Gradient** skalární veličiny

  Kartézské souřadnice:

$$\mathrm{grad}\, U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right)$$

  Válcové souřadnice:

$$\mathrm{grad}\, U = \left(\frac{\partial U}{\partial r}, \frac{1}{r}\frac{\partial U}{\partial \varphi}, \frac{\partial U}{\partial z}\right)$$

  Sférické souřadnice (pořadí složek je r, θ, φ):

$$\mathrm{grad}\, U = \left(\frac{\partial U}{\partial r}, \frac{1}{r}\frac{\partial U}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}\right)$$

- **Divergence** vektorové veličiny

  Kartézské souřadnice:

$$\mathrm{div}\,\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

  Válcové souřadnice:

$$\mathrm{div}\,\mathbf{v} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}$$

  Sférické souřadnice:

$$\mathrm{div}\,\mathbf{v} = \frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}$$

- **Rotace** vektorové veličiny

  Kartézské souřadnice:

$$\mathrm{rot}\,\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Válcové souřadnice:

$$\operatorname{rot}\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}, \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \frac{1}{r}\frac{\partial(rv_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\varphi}\right)$$

Sférické souřadnice (pořadí složek je r, θ, φ):

$$\operatorname{rot}\mathbf{v} = \left(\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta v_\varphi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\right], \frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{1}{r}\frac{\partial(rv_\varphi)}{\partial r}, \frac{1}{r}\frac{\partial(rv_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)$$

- **Tenzor deformace** vyjádřený pomocí vektoru posunutí $\mathbf{u} = (u_x, u_y, u_z)$

Kartézské souřadnice:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

Válcové souřadnice:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$2\varepsilon_{\varphi z} = \frac{1}{r}\frac{\partial u_z}{\partial\varphi} + \frac{\partial u_\varphi}{\partial z}, \quad 2\varepsilon_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad 2\varepsilon_{r\varphi} = \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\varphi}$$

Sférické souřadnice:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \quad \varepsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r},$$

$$2\varepsilon_{\theta\varphi} = \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial\theta} - u_\varphi\cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}, \quad 2\varepsilon_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}, \quad 2\varepsilon_{r\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}$$

- **Identity** pro vektorová pole

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \frac{1}{2}\operatorname{grad} v^2 - \mathbf{v}\times\operatorname{rot}\mathbf{v}$$

$$\Delta\mathbf{v} = \operatorname{grad}\operatorname{div}\mathbf{v} - \operatorname{rot}\operatorname{rot}\mathbf{v}$$

$$\operatorname{rot}\operatorname{grad}\varphi = \mathbf{0}$$

$$\operatorname{div}\operatorname{rot}\mathbf{v} = 0$$

- **Rovnice rovnováhy izotropního tělesa**

$$\operatorname{grad}\operatorname{div}\mathbf{u} - \frac{1-2\sigma}{2(1-\sigma)}\operatorname{rot}\operatorname{rot}\mathbf{u} = -\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\,\mathbf{f}$$

kde $\mathbf{u}$ je vektor posunutí při deformaci a $\mathbf{f}$ je objemová hustota objemových sil.

Gradient skalární funkce

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\,\vec{e}_x + \frac{\partial f}{\partial y}\,\vec{e}_y + \frac{\partial f}{\partial z}\,\vec{e}_z$$

$$\nabla f(\varrho,\varphi,z) = \frac{\partial f}{\partial \varrho}\,\vec{e}_\varrho + \frac{1}{\varrho}\frac{\partial f}{\partial \varphi}\,\vec{e}_\varphi + \frac{\partial f}{\partial z}\,\vec{e}_z$$

$$\nabla f(\varrho,\vartheta,\varphi) = \frac{\partial f}{\partial \varrho}\,\vec{e}_\varrho + \frac{1}{\varrho}\frac{\partial f}{\partial \vartheta}\,\vec{e}_\vartheta + \frac{1}{\varrho\sin\vartheta}\frac{\partial f}{\partial \varphi}\,\vec{e}_\varphi$$

Divergence vektorové funkce

$$\vec{\nabla}\cdot\vec{A}(x,y,z) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\cdot\vec{A}(\varrho,\varphi,z) = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}(\varrho A_\varrho) + \frac{1}{\varrho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\cdot\vec{A}(\varrho,\vartheta,\varphi) = \frac{1}{\varrho^2}\frac{\partial}{\partial \varrho}(\varrho^2 A_\varrho) + \frac{1}{\varrho\sin\vartheta}\frac{\partial}{\partial \vartheta}(\sin\vartheta A_\vartheta) + \frac{1}{\varrho\sin\vartheta}\frac{\partial A_\varphi}{\partial \varphi}$$

Rotace vektorové funkce

$$\vec{\nabla}\times\vec{A}(x,y,z) = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\vec{e}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\vec{e}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\vec{e}_z$$

$$\vec{\nabla}\times\vec{A}(\varrho,\varphi,z) = \left[\frac{1}{\varrho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\vec{e}_\varrho + \left[\frac{\partial A_\varrho}{\partial z} - \frac{\partial A_z}{\partial \varrho}\right]\vec{e}_\varphi + \frac{1}{\varrho}\left[\frac{\partial}{\partial \varrho}(\varrho A_\varphi) - \frac{\partial A_\varrho}{\partial \varphi}\right]\vec{e}_z$$

$$\vec{\nabla}\times\vec{A}(\varrho,\vartheta,\varphi) = \frac{1}{\varrho\sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(\sin\vartheta A_\varphi) - \frac{\partial A_\vartheta}{\partial \varphi}\right]\vec{e}_\varrho + \frac{1}{\varrho}\left[\frac{1}{\sin\vartheta}\frac{\partial A_\varrho}{\partial \varphi} - \frac{\partial}{\partial \varrho}(\varrho A_\varphi)\right]\vec{e}_\vartheta +$$
$$+\frac{1}{\varrho}\left[\frac{\partial}{\partial \varrho}(\varrho A_\vartheta) - \frac{\partial A_\varrho}{\partial \vartheta}\right]\vec{e}_\varphi$$

Laplacián skalární funkce

$$\Delta f(x,y,z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta f(\varrho,\varphi,z) = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(\varrho\frac{\partial f}{\partial \varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta f(\varrho,\vartheta,\varphi) = \frac{1}{\varrho^2}\frac{\partial}{\partial \varrho}\left(\varrho^2\frac{\partial f}{\partial \varrho}\right) + \frac{1}{\varrho^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial f}{\partial \vartheta}\right) + \frac{1}{\varrho^2\sin^2\vartheta}\frac{\partial^2 f}{\partial \varphi^2}$$

Laplacián vektorové funkce

$$\Delta\vec{A}(x,y,z) = \Delta A_x\,\vec{e}_x + \Delta A_y\,\vec{e}_y + \Delta A_z\,\vec{e}_z$$

$$\Delta\vec{A}(\varrho,\varphi,z) = \left(\Delta A_\varrho - \frac{A_\varrho}{\varrho^2} - \frac{2}{\varrho^2}\frac{\partial A_\varphi}{\partial \varphi}\right)\vec{e}_\varrho + \left(\Delta A_\varphi - \frac{A_\varphi}{\varrho^2} + \frac{2}{\varrho^2}\frac{\partial A_\varrho}{\partial \varphi}\right)\vec{e}_\varphi + \Delta A_z\,\vec{e}_z$$

$$\Delta\vec{A}(\varrho,\vartheta,\varphi) = \left[\Delta A - \varrho - \frac{2A_\varrho}{\varrho^2} - \frac{2}{\varrho^2\sin\vartheta}\frac{\partial}{\partial \vartheta}(\sin\vartheta A_\vartheta) - \frac{2}{\varrho^2\sin\vartheta}\frac{\partial A_\varphi}{\partial \varphi}\right]\vec{e}_\varrho +$$
$$+\left[\Delta A_\vartheta - \frac{A_\vartheta}{\varrho^2\sin^2\vartheta} + \frac{2}{\varrho^2}\frac{\partial A_\varrho}{\partial \vartheta} - \frac{2\cos\vartheta}{\varrho^2\sin^2\vartheta}\frac{\partial A_\varphi}{\partial \varphi}\right]\vec{e}_\vartheta +$$
$$+\left[\Delta A_\varphi - \frac{A_\varphi}{\varrho^2\sin^2\vartheta} + \frac{2}{\varrho^2\sin\vartheta}\frac{\partial A_\varrho}{\partial \varphi} + \frac{2\cos\vartheta}{\varrho^2\sin^2\vartheta}\frac{\partial A_\vartheta}{\partial \varphi}\right]\vec{e}_\varphi$$

$(\vec{v} \cdot \nabla)\vec{v}$ v cylindrických souřadnicích.

Složka ve směru r

$$v_r \frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi^2}{r} + v_z \frac{\partial v_r}{\partial z}$$

Složka ve směru φ

$$v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r v_\varphi}{r} + v_z \frac{\partial v_\varphi}{\partial z}$$

Složka ve směru z

$$v_r \frac{\partial v_z}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_z}{\partial \varphi} + v_z \frac{\partial v_z}{\partial z}$$